Closing Fri: 2.1, 2.2, 2.3

Closing next Tue: 2.5-6

Closing next Fri: 2.7, 2.7-8

Today: Finish 2.3, start 2.5.

Entry Task: Find the limits

1. $\lim\limits_{x\to 0} \dfrac{x^2+6x+5}{x+1} = \dfrac{5}{1} = \boxed{5}$

2. $\lim\limits_{x\to -1} \dfrac{x^2+6x+5}{x+1} =$ $\frac{0}{0}$ ← ALGEBRA NEEDED

$= \lim\limits_{x\to -1} \dfrac{\cancel{(x+1)}(x+5)}{\cancel{(x+1)}}$

$= \lim\limits_{x\to -1} x+5 = \boxed{4}$

3. $\lim\limits_{x\to 0^+} \dfrac{e^x - \sin\left(x+\frac{\pi}{2}\right)+3}{x(x-4)} =$

$\frac{1-1+3=3}{0} \quad \frac{3}{0}$

↑ EITHER $+\infty$ OR $-\infty$

NUMERATOR → 3 ← (POSITIVE)

DENOMINATOR → 0 ←

FOR X SLIGHTLY BIGGER THAN 0

$\underset{+}{x}\underbrace{(x-4)}_{-}$ WILL BE NEGATIVE

THUS, $\boxed{-\infty}$

4. $\lim\limits_{x\to 2} \dfrac{\frac{1}{x+1}-\frac{1}{3}}{x-2} =$ $\frac{0}{0}$ ← ALGEBRA NEEDED

$\lim\limits_{x\to 2} \dfrac{\left(\frac{1}{x+1}-\frac{1}{3}\right)\cdot 3\cdot(x+1)}{(x-2)\cdot 3\cdot(x+1)}$

$= \lim\limits_{x\to 2} \dfrac{3-(x+1)}{3(x-2)(x+1)}$

$= \lim\limits_{x\to 2} \dfrac{3-x-1}{3(x-2)(x+1)} = \lim\limits_{x\to 2} \dfrac{2-x}{3(x-2)(x+1)}$

$= \lim\limits_{x\to 2} \dfrac{-\cancel{(x-2)}}{3\cancel{(x-2)}(x+1)} = \dfrac{-1}{3(2+1)} = \boxed{-\frac{1}{9}}$

5. $\lim_{h\to 0} \frac{(6+h)^2 - 36}{h} = \frac{0}{0}$ ← ALGEBRA

$$= \lim_{h\to 0} \frac{\cancel{36} + 12h + h^2 - \cancel{36}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(12+h)}{\cancel{h}}$$

$$= 12 + 0 = \boxed{12}$$

6. $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \frac{0}{0}$

THE CONJUGATE OF $\sqrt{x} - 2$ IS $\sqrt{x} + 2$ AND NOTE

$$(\sqrt{x} - 2)(\sqrt{x} + 2) = x - 4$$

$$\lim_{x\to 4} \frac{(x-4)}{(\sqrt{x}-2)} \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}}$$

$$= \sqrt{4} + 2 = \boxed{4}$$

Squeeze Thm:

If the following hold:

(1) $g(x) \le f(x) \le h(x)$ near $x = a$

(2) $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$

then

$$\lim_{x \to a} f(x) = L$$

Example: Find

$$\lim_{x \to 0} x^2 \cos\left(\frac{10}{x}\right) = 0$$

WE KNOW

$$-1 \le \cos\left(\frac{10}{x}\right) \le 1$$

IS ALWAYS TRUE.

mult. by x^2

THUS,

$$-x^2 \le x^2 \cos\left(\frac{10}{x}\right) \le x^2$$

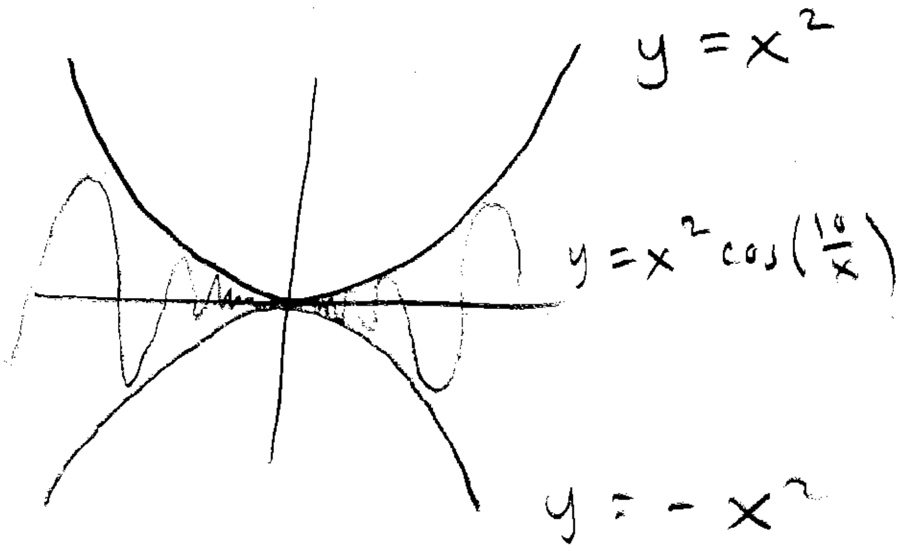
IS ALWAYS TRUE.

SINCE $\lim_{x \to 0} -x^2 = 0$ AND

$\lim_{x \to 0} x^2 = 0$

WE HAVE

$$\lim_{x \to 0} x^2 \cos\left(\frac{10}{x}\right) = 0$$



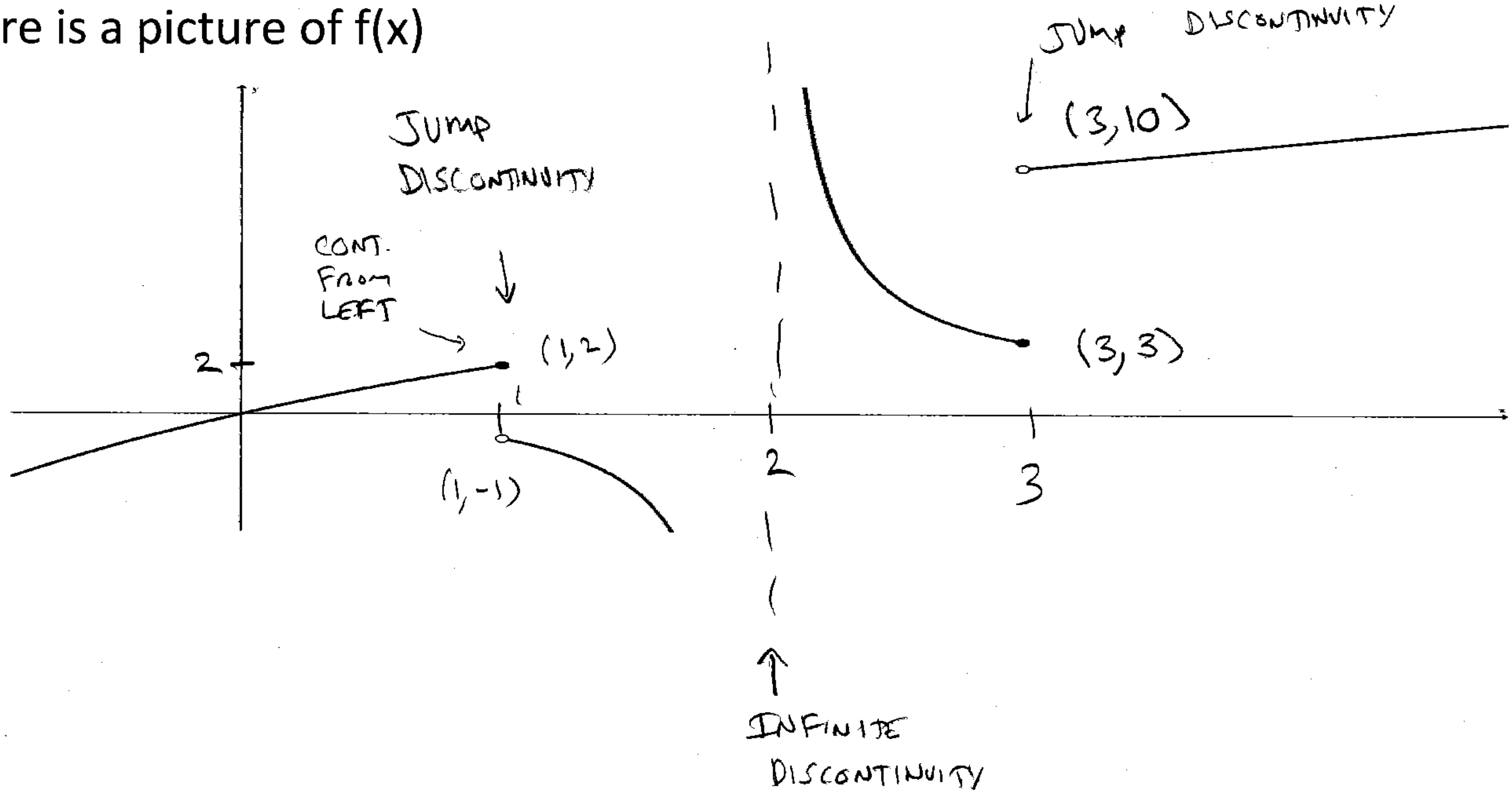
Multipart functions (again):

Example: Find the limits

$$f(x) = \begin{cases} \dfrac{12x}{x+5} & \text{, if } x \le 1; \\ \dfrac{x}{x-2} & \text{, if } 1 < x \le 3 \text{ and } x \ne 2; \\ \dfrac{x^2+4x-21}{x-3} & \text{, if } x > 3. \end{cases}$$

1. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{12x}{x+5} = \frac{12}{6} = 2$
2. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{x-2} = \frac{1}{-1} = -1$
3. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x}{x-2} = -\infty$
4. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x}{x-2} = +\infty$
5. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x}{x-2} = \frac{3}{1} = 3$
6. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2+4x-21}{x-3} = \lim_{x\to 3^+} \frac{(x+7)(x-3)}{(x-3)} = 10$

Here is a picture of f(x)



2.5 Continuity

A function, $f(x)$, is **continuous at x = a** if

$$\lim_{x \to a} f(x) = f(a)$$

this implies three things

(1). f(a) is defined,

(2). $\lim_{x \to a} f(x)$ exists and is finite, and

(3). (1) and (2) are the same!

Continuous from the left

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuous from the right

$$\lim_{x \to a^+} f(x) = f(a)$$

Casually, we might say a function is continuous at *x* = *a* if you can draw the graph across *x* = *a* point without picking up your pencil.

The “standard” precalculus functions are **continuous everywhere they are defined**:

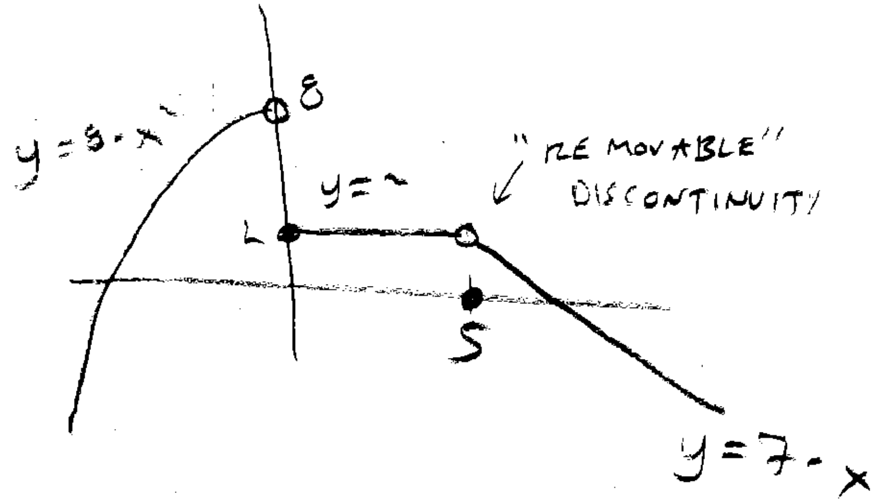
polynomials → defined everywhere
sin(x), cos(x) → defined everywhere
e^x → defined everywhere
odd roots → defined everywhere
$\tan^{-1}(x)$ → defined everywhere
Rational Functions → for denom $\neq 0$
Even Roots → under radical ≥ 0
ln(x) → for $x > 0$
tan(x) → not at $x = \pm k\pi/2$
$\sin^{-1}(x)$, $\cos^{-1}(x)$ → for $-1 \leq x \leq 1$

Example:

$$g(x) = \begin{cases} 8 - x^2 & \text{, if } x < 0; \\ 2 & \text{, if } 0 \leq x < 5; \\ 0 & \text{, if } x = 5; \\ 7 - x & \text{, if } x > 5. \end{cases}$$

Does g(x) have any discontinuities?

If so, where?



SINCE ALL FUNCTIONS ARE POLYNOMIALS (OR CONSTANTS), THE ONLY DISCONTINUITIES CAN OCCUR AT THE "ENDPOINTS" OF THE GIVEN DOMAINS

$\boxed{x = 0\ ?}$

$$\left.\begin{array}{l} \lim_{x\to 0^-} g(x) = 8 \\ \lim_{x\to 0^+} g(x) = 2 \\ g(0) = 2 \end{array}\right\}$$ NOT THE SAME

$\boxed{x = 5\ ?}$

$$\left.\begin{array}{l} \lim_{x\to 5^-} g(x) = 2 \\ \lim_{x\to 5^-} g(x) = 7 - 5 = 2 \\ g(5) = 0 \end{array}\right\}$$ NOT THE SAME

$\boxed{x = 0}$ & $\boxed{x = 5}$

Example:

$$h(x) = \begin{cases} ax^2 + 6 & , \text{if } x < 1; \\ b & , \text{if } x = 1; \\ \dfrac{x+49}{x+a} & , \text{if } x > 1. \end{cases}$$

Find the values of *a* and *b* that will make $h(x)$ continuous *everywhere*.

WE ARE WORRIED ABOUT

$x = 1$ AND $x = -a$.

$\boxed{x = 1 ?}$

$$\left.\begin{aligned} \lim_{x \to 1^-} h(x) &= a + 6 \\ \lim_{x \to 1^+} h(x) &= \frac{50}{1+a} \\ h(1) &= b \end{aligned}\right\} \text{WANT ALL THE SAME!}$$

$$\left.\begin{aligned} a + 6 &= \frac{50}{1+a} \\ \Rightarrow (a+6)(1+a) &= 50 \\ \Rightarrow a^2 + 7a + 6 &= 50 \\ \Rightarrow a^2 + 7a - 44 &= 0 \\ (a+11)(a-4) &= 0 \end{aligned}\right\} \begin{aligned} &a = 4 \Rightarrow b = 10 \\ &\quad \text{OR} \\ &a = -11 \Rightarrow b = -5 \end{aligned}$$

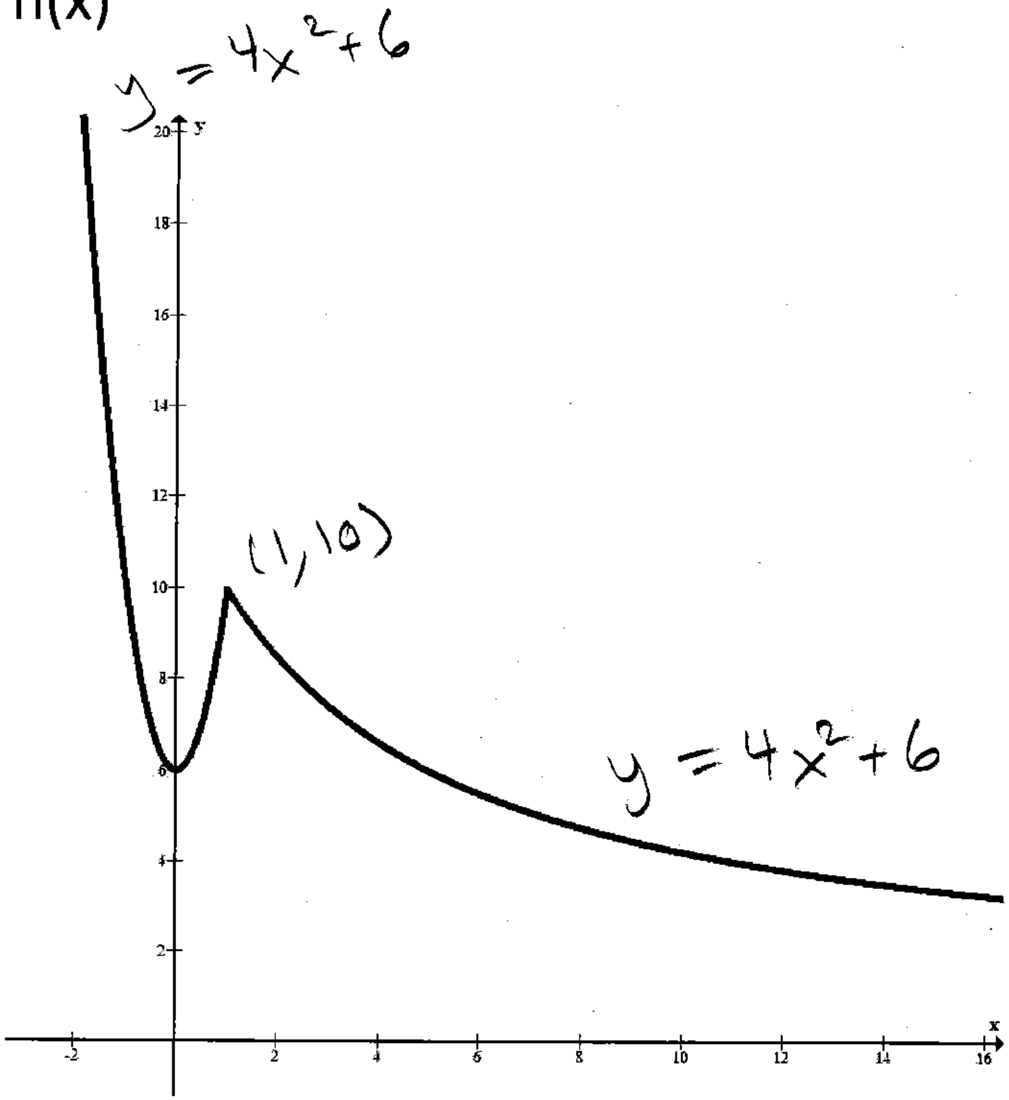
BUT

$$h(x) = \begin{cases} -11x^2 + 6 & , \text{if } x < 1; \\ -5 & , \text{if } x = 1; \\ \dfrac{x+49}{x-11} & , \text{if } x > 1. \end{cases}$$

CANNOT BE THE ANSWER BECAUSE IT IS DISCONTINUOUS AT $x = 11$ WHICH IS IN THIS DOMAIN

$$\boxed{h(x) = \begin{cases} 4x^2 + 6, & \text{if } x < 1; \\ 10, & \text{if } x = 1; \\ \dfrac{x+49}{x+4}, & \text{if } x > 1 \end{cases}}$$

h(x)



Theorem:

If $f(x)$ is continuous at $x = b$, and

$$\lim_{x \to a} g(x) = b$$

then

$$\lim_{x \to a} f\big(g(x)\big) = f(b).$$

Example:

Find

$$\lim_{x \to 9} \ln\left(\frac{\sqrt{x} - 3}{x - 9}\right)$$

$$= \ln\left(\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}\right)$$

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$$

$$= \lim_{x \to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \frac{1}{3+3} = \boxed{\frac{1}{6}}$$